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CP Violation and Unphysical Phases

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Contents

1. Introduction

We often encounter the terms "mixing phase" and "decay phase" in the analysis of CP asymmetries. These terms are misleading, because the splitting into a phase from the mixing amplitude and a phase from the decay amplitude is arbitrary and a matter of convention.

When many years ago I first discovered the existence of many unphysical phases connected with the CP transformation and the CKM matrix, I thought that I, a meek experimentalist, had missed the way to properly fix at least some of these arbitrary phases. I am therefore infinitely indebted to Helen Quinn, who confirmed that all this arbitrariness is indeed true.

2. Physical and Unphysical Phases

2.1 The Unitary CKM Matrix

The charged current weak interactions responsible for flavour changes are described in the Standard Model by the couplings $gW^\mu J^{cc}_\mu$ of the W boson to the current

$$J^{cc}_\mu = \begin{pmatrix} \overline{\nu}_e \\ \overline{\nu}_\mu \\ \overline{\nu}_\tau \end{pmatrix} \gamma_\mu \frac{1-\gamma_5}{2} \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix} + \sum_{r,g,b} \begin{pmatrix} \bar{u} \\ \bar{c} \\ \bar{t} \end{pmatrix} \gamma_\mu \frac{1-\gamma_5}{2} \mathbf{V} \cdot \begin{pmatrix} d \\ s \\ b \end{pmatrix} \tag{2.1}$$

with a non-trivial transformation matrix $\mathbf{V}$ in the quark sector, the Cabibbo–Kobayashi–Maskawa (CKM) Matrix [1,2]:

$$\mathbf{V} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

A coupling via a scalar boson would allow a general 3×3 coupling matrix. However, local gauge invariance which is realized via the vector gauge bosons $W^\pm$ requires that one universal coupling constant connects the triplet of up-type quarks with the triplet of down-type quarks. The only complication permitted is a unitary transformation to another basis of states, which is accomplished by the CKM matrix.

In (2.1), the quark states used are mass eigenstates, and the CKM matrix can be completely represented by Yukawa couplings of the quark fields to the scalar Higgs field. The corresponding PMNS matrix in the lepton sector has been absorbed in the neutrino fields $(\overline{\nu}_e, \overline{\nu}_\mu, \overline{\nu}_\tau)$ which are not mass eigenstates.

From the 9 real parameters of a general unitary matrix, 5 can be absorbed in 1 global phase, 2 relative phases between u, c, t and 2 relative phases between d, s, b which are all subject to **convention** and in principle **unobservable**. If two quarks within one of these two groups were degenerate in mass, even the sixth phase could be removed by redefining the basis in their two-dimensional subspace.

Rephasing may be accomplished by applying a phase factor to every row and column:

$$V_{jk} \to e^{i(\phi_j - \phi_k)} V_{jk} \tag{2.2}$$

Note that $j = u, c, t$, $k = d, s, b$, and the six numbers $\phi_u, \phi_c, \phi_t, \phi_d, \phi_s, \phi_b$ represent only five independent phases in the CKM matrix, since different sets of $\{\phi_j, \phi_k\}$ yield the same result. Any product where each row and column enters once as V_{ij} and once via a complex conjugate V^*_{kl} like $V_{ij}V_{kl}V^*_{il}V^*_{kj}$ is **invariant** under the transformation (2.2). This implies that observable phases must always correspond to similar products of CKM matrix elements with equal numbers of V and V^* factors and appropriate combination of indices.

Removing as much unphysical phases as possible, the CKM matrix is described by **4 real parameters**, where only one is a phase parameter, while the other three are rotation angles in flavour space. The physical phase is not one unique number due to the arbitrary choice of the unphysical phases.

Unambiguous representations of this phase as the angles of unitarity triangles will be discussed below. The standard parametrization [3] (first proposed in [4], notation follows [5]) uses a choice of phases, that leave V_{ud} and V_{cb} real:

$$\mathbf{V} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta_{13}} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta_{13}} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23}-c_{12}s_{13}s_{23}e^{i\delta_{13}} & c_{12}c_{23}-s_{12}s_{13}s_{23}e^{i\delta_{13}} & c_{13}s_{23} \\ s_{12}s_{23}-c_{12}s_{13}c_{23}e^{i\delta_{13}} & -c_{12}s_{23}-s_{12}s_{13}c_{23}e^{i\delta_{13}} & c_{13}c_{23} \end{pmatrix} \tag{2.3}$$

with $c_{ij} = \cos\theta_{ij}$, $s_{ij} = \sin\theta_{ij}$, and $s_{ij} > 0$, $c_{ij} > 0$ $(0 \leq \theta_{ij} \leq \pi/2)$. The angle $\theta_C = \theta_{12}$ is the Cabibbo-angle [1].

A convenient substitution[1] is $s_{12} = \lambda$, $s_{23} = A\lambda^2$, $s_{13}\sin\delta_{13} = A\lambda^3\eta$, and $s_{13}\cos\delta_{13} = A\lambda^3\rho$ [6], which reflects the apparent hierarchy in the size of mixing angles via orders of a parameter λ. This leads to

$$\mathbf{V} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{1-A^2\lambda^4} & A\lambda^2 \\ 0 & -A\lambda^2 & \sqrt{1-A^2\lambda^4} \end{pmatrix} \cdot$$
$$\cdot \begin{pmatrix} \sqrt{1-A^2\lambda^6(\rho^2+\eta^2)} & 0 & A\lambda^3(\rho-i\eta) \\ 0 & 1 & 0 \\ -A\lambda^3(\rho+i\eta) & 0 & \sqrt{1-A^2\lambda^6(\rho^2+\eta^2)} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{1-\lambda^2} & \lambda & 0 \\ -\lambda & \sqrt{1-\lambda^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1-\frac{\lambda^2}{2}-\frac{\lambda^4}{8} & \lambda & A\lambda^3(\rho-i\eta) \\ -\lambda-A^2\lambda^5(\rho+i\eta-\frac{1}{2}) & 1-\frac{\lambda^2}{2}-(\frac{1}{8}+\frac{A}{2})\lambda^4 & A\lambda^2 \\ A\lambda^3[1-(\rho+i\eta)(1-\frac{\lambda^2}{2})] & -A\lambda^2-A\lambda^4(\rho+i\eta-\frac{1}{2}) & 1-\frac{1}{2}A^2\lambda^4 \end{pmatrix} + \mathcal{O}(\lambda^6) \tag{2.4}$$

and agrees to $\mathcal{O}(\lambda^3)$ with the Wolfenstein approximation[2] [7]:

$$\mathbf{V} = \begin{pmatrix} 1-\frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho-i\eta+\frac{i}{2}\eta\lambda^2) \\ -\lambda & 1-\frac{\lambda^2}{2}-i\eta A^2\lambda^4 & A\lambda^2(1+i\eta\lambda^2) \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} \tag{2.5}$$

$$\mathbf{V} \approx \begin{pmatrix} 1-\frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho-i\eta) \\ -\lambda & 1-\frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} \tag{2.6}$$

Equation (2.4) is more convenient [8] in higher orders than the original proposal of Wolfenstein, or an exact parametrization [9] using the Wolfenstein parameters.

[1] An equivalent choice is $\lambda = s_{12}c_{13}$ which leads to the same parametrization to $\mathcal{O}(\lambda^5)$.
[2] using $\lambda := V_{us} = s_{12}c_{13}$, $A\lambda^2 := V_{cb} = s_{23}c_{13}$ and unitarity at $\mathcal{O}(\lambda^3)$ to define ρ, η, and using a phase convention that leaves V_{ud}, V_{us}, V_{tb}, V_{ts}, and especially V_{cd} real to $\mathcal{O}(\lambda^5)$.

2.1.1 Unitarity Triangles

If nature provides us with just these three families of fermions, unitarity requires the following 12 conditions to be fulfilled:

rows 1×1, uu	$\lvert V_{ud}\rvert^2 + \lvert V_{us}\rvert^2 + \lvert V_{ub}\rvert^2 = 1$	(2.7a)
rows 2×2, cc	$\lvert V_{cd}\rvert^2 + \lvert V_{cs}\rvert^2 + \lvert V_{cb}\rvert^2 = 1$	(2.7b)
rows 3×3, tt	$\lvert V_{td}\rvert^2 + \lvert V_{ts}\rvert^2 + \lvert V_{tb}\rvert^2 = 1$	(2.7c)
columns 1×1, dd	$\lvert V_{ud}\rvert^2 + \lvert V_{cd}\rvert^2 + \lvert V_{td}\rvert^2 = 1$	(2.7d)
columns 2×2, ss	$\lvert V_{us}\rvert^2 + \lvert V_{cs}\rvert^2 + \lvert V_{ts}\rvert^2 = 1$	(2.7e)
columns 3×3, bb	$\lvert V_{ub}\rvert^2 + \lvert V_{cb}\rvert^2 + \lvert V_{tb}\rvert^2 = 1$	(2.7f)
rows 1×2, cu	$V_{ud}^*V_{cd} + V_{us}^*V_{cs} + V_{ub}^*V_{cb} = 0$	(2.7g)
rows 1×3, tu	$V_{ud}^*V_{td} + V_{us}^*V_{ts} + V_{ub}^*V_{tb} = 0$	(2.7h)
rows 2×3, tc	$V_{cd}^*V_{td} + V_{cs}^*V_{ts} + V_{cb}^*V_{tb} = 0$	(2.7i)
columns 1×2, sd	$V_{ud}V_{us}^* + V_{cd}V_{cs}^* + V_{td}V_{ts}^* = 0$	(2.7j)
columns 1×3, bd	$V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0$	(2.7k)
columns 2×3, bs	$V_{us}V_{ub}^* + V_{cs}V_{cb}^* + V_{ts}V_{tb}^* = 0$	(2.7l)

The conditions a–f are redundant, three are sufficient to establish unitarity together with g–l. An arbitrary phase for the whole matrix cancels in $\mathbf{V^+V}$. A phase common to all elements in a line (column), corresponding to arbitrary phases between u, c, t (d, s, b) will vanish in eqns. 2.7j–l (2.7g–i) and become a common factor in eqns. 2.7g–i (2.7j–l).

Dividing (2.7k) by $A\lambda^3 \approx -V_{cd}V_{cb}^*$ yields the unitarity triangle[3] as shown in figure 2.1a. In the Wolfenstein approximation, it corresponds to

$$(\rho + i\eta) - 1 + (1 - \rho - i\eta) = 0 \tag{2.8}$$

A second one from (2.7h) is shown in figure 2.1b. Dividing by $A\lambda^3 \approx -V_{us}^*V_{ts}$ and using the approximation $V_{ud} \approx 1$ gives the same triangle (2.8). A closer look, however, reveals slightly different lengths and angles to $\mathcal{O}(\lambda^2)$.

The angles[4] of the unitarity triangles bd and tu (2.7k and h) in figure 2.1 are defined by[5]

$$\begin{aligned}
e^{i\alpha} &= -\frac{V_{td}V_{ub}V_{ud}^*V_{tb}^*}{|V_{td}V_{ub}V_{ud}V_{tb}|}\\
e^{i\beta} &= -\frac{V_{td}^*V_{cb}^*V_{cd}V_{tb}}{|V_{td}V_{cb}V_{cd}V_{tb}|} \approx e^{i\beta'} = -\frac{V_{td}^*V_{us}^*V_{ts}V_{ud}}{|V_{td}V_{us}V_{ts}V_{ud}|}\\
e^{i\gamma} &= -\frac{V_{ub}^*V_{cd}^*V_{cb}V_{ud}}{|V_{ub}V_{cd}V_{cb}V_{ud}|} \approx e^{i\gamma'} = -\frac{V_{ub}^*V_{ts}^*V_{us}V_{tb}}{|V_{ub}V_{ts}V_{us}V_{tb}|}
\end{aligned}$$

These are **rephasing invariant** expressions, hence the angles resemble physical quantities independent of the CKM parametrization. It was first emphasized by Jarlskog [12], that CP violation can be described via a rephasing invariant quantity

$$J = \pm\mathcal{I}m\, V_{ij}V_{kl}V_{il}^*V_{kj}^* \approx A^2\lambda^6\eta$$

which is up to a sign independent of i, j, k, l, provided $i \neq k$, $j \neq l$.

$$J = \mathcal{I}m(V_{ud}V_{cs}V_{us}^*V_{cd}^*) = -\mathcal{I}m(V_{ud}V_{cb}V_{ub}^*V_{cd}^*) = -\mathcal{I}m(V_{ud}V_{ts}V_{us}^*V_{td}^*)$$

[3] This geometric interpretation has been pointed out by Bjorken ∼1986; its first documentation in printed form is in ref. 10 and more general in ref. 11.

[4] Another naming convention is $\phi_1 = \beta$, $\phi_2 = \alpha$ and $\phi_3 = \gamma$.

[5] In the complex plane, the angle $\alpha - \beta$ between two vectors $A = ae^{i\alpha}$ and $B = be^{i\beta}$ is given by $e^{i(\alpha-\beta)} = AB^*/|AB|$ and $\sin(\alpha - \beta) = \mathcal{I}m(AB^*)/|AB| = (AB^* - A^*B)/(2i|AB|)$.

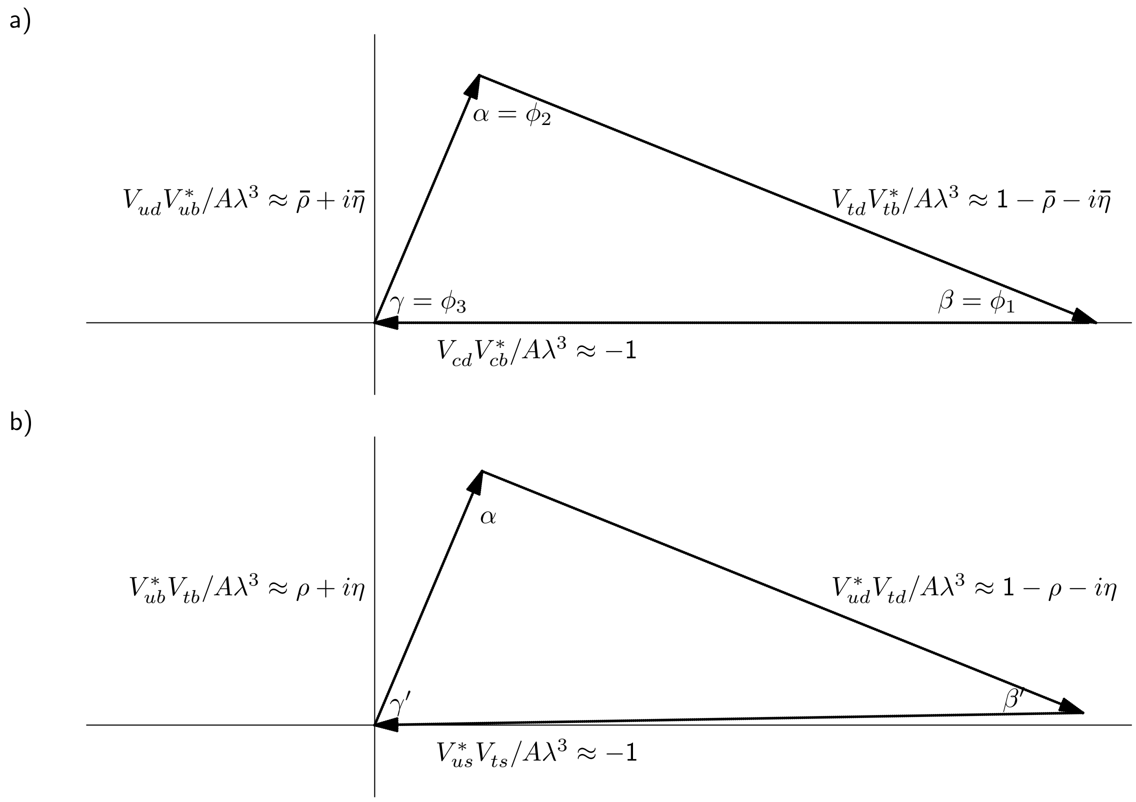

Fig. 2.1 Unitarity triangles bd and tu in the complex plane, corresponding to **a**: (2.7k) and **b**: (2.7h), respectively. Up to corrections of $\mathcal{O}(\lambda^4)$ the top points are (ρ, η) in **(b)**, but $(\bar\rho = [1 - \frac{\lambda^2}{2}]\rho, \bar\eta = [1 - \frac{\lambda^2}{2}]\eta)$ in **(a)**, and the rightmost points are $(1, 0)$ in **(a)**, but $(1 - \lambda^2[\frac{1}{2} - \rho], \lambda^2\eta)$ in **(b)**. The angles are related via $\gamma - \gamma' = \beta' - \beta \approx \beta_s \approx \lambda^2\eta$. Changing the phase convention for the CKM matrix will rotate the triangles in the complex plane, but their shape is invariant under those transformations. To avoid this, we may use $-V_{cd}V_{cb}^*$ instead of $A\lambda^3$ as scale factor, which makes the baseline of the triangle (a) exactly -1.

$$\begin{aligned}
&= \mathcal{I}m(V_{ud}V_{tb}V_{ub}^*V_{td}^*) &&= -\mathcal{I}m(V_{us}V_{cd}V_{ud}^*V_{cs}^*) &&= \mathcal{I}m(V_{us}V_{cb}V_{ub}^*V_{cs}^*)\\
&= \mathcal{I}m(V_{us}V_{td}V_{ud}^*V_{ts}^*) &&= -\mathcal{I}m(V_{us}V_{tb}V_{ub}^*V_{ts}^*) &&= \mathcal{I}m(V_{ub}V_{cd}V_{ud}^*V_{cb}^*)\\
&= -\mathcal{I}m(V_{ub}V_{cs}V_{us}^*V_{cb}^*) &&= -\mathcal{I}m(V_{ub}V_{td}V_{ud}^*V_{tb}^*) &&= \mathcal{I}m(V_{ub}V_{ts}V_{us}^*V_{tb}^*)\\
&= \mathcal{I}m(V_{cd}V_{ts}V_{cs}^*V_{td}^*) &&= -\mathcal{I}m(V_{cd}V_{tb}V_{cb}^*V_{td}^*) &&= -\mathcal{I}m(V_{cs}V_{td}V_{cd}^*V_{ts}^*)\\
&= \mathcal{I}m(V_{cs}V_{tb}V_{cb}^*V_{ts}^*) &&= \mathcal{I}m(V_{cb}V_{td}V_{cd}^*V_{tb}^*) &&= -\mathcal{I}m(V_{cb}V_{ts}V_{cs}^*V_{tb}^*)
\end{aligned}$$

These terms are all products of the type $\mathcal{I}m\, AB^* = |A||B|\,\mathcal{I}m\, e^{i(\arg A - \arg B)} = |A||B|\sin(\arg A - \arg B)$, which is twice the area of a triangle in the complex plane with sides A and B. The A and B here are sides of a unitarity triangle. The equality of these terms is easily seen, e. g. for the last line replacing d with s is equivalent to applying the unitarity condition (2.7i)

$$V_{td}V_{cd}^* = -V_{ts}V_{cs}^* - V_{tb}V_{cb}^*$$

which yields

$$\mathcal{I}m(V_{cb}V_{td}V_{cd}^*V_{tb}^*) = -\mathcal{I}m(V_{cb}V_{ts}V_{cs}^*V_{tb}^*) - \mathcal{I}m(V_{cb}V_{tb}V_{cb}^*V_{tb}^*)$$

and the last argument is real, i. e. $\mathcal{I}m(V_{cb}V_{tb}V_{cb}^*V_{tb}^*) = \mathcal{I}m\,|V_{cb}|^2|V_{tb}|^2 = 0$. Hence the areas of all six unitarity triangles defined by (2.7g–l) are equal and have the value $J/2$. This corresponds to an area $\approx \eta/2$ for the ones in figure 2.1, since their sides have been reduced by the factor $A\lambda^3$. As will be shown below, CP violating observables are typically proportional to the sine of the angles in unitarity triangles,

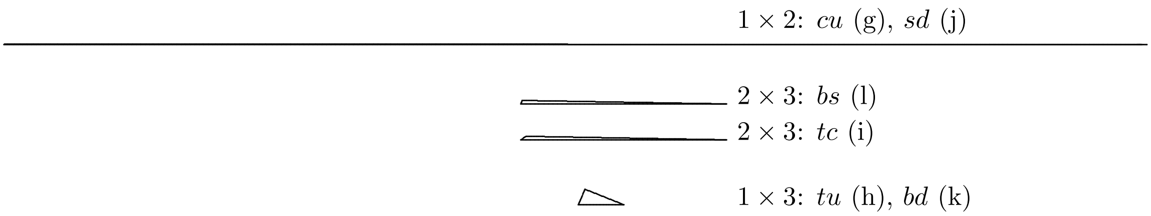

Fig. 2.2 The three types of unitarity triangles, approximately to scale. Their areas are equal.

like

$$\sin\gamma = \mathcal{I}m\, e^{i\gamma} = -\frac{\mathcal{I}m(V_{ub}^* V_{cd}^* V_{cb} V_{ud})}{|V_{ub}V_{cd}V_{cb}V_{ud}|} = -\frac{J}{|V_{ub}V_{cd}V_{cb}V_{ud}|}$$

and vanish for $J = 0$, i. e. if all triangles collapse into lines. If the non-trivial phase in the CKM matrix is 0 or π, the parameter η is 0 and hence $J = 0$. This would also be the case if two quarks of a given charge had the same mass, since then a rotation between these two flavours could be chosen that removes the phase factors, as can be seen in (2.3) where $\theta_{13} = 0$ would remove all terms with the phase δ_{13}.

All six unitarity triangles are shown approximately to scale in figure 2.2. Their angles can be determined using the standard parametrization (2.3) in a rewritten form

$$\mathbf{V} = \begin{pmatrix} |V_{ud}| & |V_{us}| & |V_{ub}|e^{-i\tilde{\gamma}} \\ -|V_{cd}|e^{i\tilde{\phi}_4} & |V_{cs}|e^{-i\tilde{\phi}_6} & |V_{cb}| \\ |V_{td}|e^{-i\tilde{\beta}} & -|V_{ts}|e^{i\tilde{\phi}_2} & |V_{tb}| \end{pmatrix} \tag{2.9}$$

with $\tilde{\gamma} \equiv \delta_{13}$. Here, absolute values and phases are given as separate factors. The angles $\tilde{\phi}_2 \approx \eta\lambda^2$, $\tilde{\phi}_4 \approx \eta A^2\lambda^4$, and $\tilde{\phi}_6 \approx \eta A^2\lambda^6$ are all positive and very small and their subscript indicates the order in λ of their magnitude. The unitarity triangles in figure 2.1 have angles

$$\begin{aligned} \beta &= \tilde{\beta} + \tilde{\phi}_4, & \beta' &= \tilde{\beta} + \tilde{\phi}_2 = \beta + \tilde{\phi}_2 - \tilde{\phi}_4 \\ \gamma &= \tilde{\gamma} - \tilde{\phi}_4, & \gamma' &= \tilde{\gamma} - \tilde{\phi}_2 = \gamma - \tilde{\phi}_2 + \tilde{\phi}_4 \\ \alpha &= \pi - \tilde{\beta} - \tilde{\gamma} = \pi - \beta - \gamma = \pi - \beta' - \gamma' \\ && \beta_s &= \tilde{\phi}_2 + \tilde{\phi}_6 \end{aligned}$$

In the Wolfenstein approximation, the unitarity relations read (all terms given to order λ^3)

$$-\lambda + \tfrac{1}{2}\lambda^3 + \lambda - \tfrac{1}{2}\lambda^3 + \quad 0 + \mathcal{O}(\lambda^5) \quad = 0 \tag{2.7g$'$}$$
$$A\lambda^3(1-\rho-i\eta) - \quad A\lambda^3 \quad + \quad A\lambda^3(\rho+i\eta) \quad = 0 \tag{2.7h$'$}$$
$$0 + \mathcal{O}(\lambda^4) - \quad A\lambda^2 \quad + \quad A\lambda^2 \quad = 0 \tag{2.7i$'$}$$
$$\lambda - \tfrac{1}{2}\lambda^3 - \lambda + \tfrac{1}{2}\lambda^3 - \quad 0 + \mathcal{O}(\lambda^5) \quad = 0 \tag{2.7j$'$}$$
$$A\lambda^3(\rho+i\eta) - \quad A\lambda^3 \quad + A\lambda^3(1-\rho-i\eta) = 0 \tag{2.7k$'$}$$
$$0 + \mathcal{O}(\lambda^4) + \quad A\lambda^2 \quad - \quad A\lambda^2 \quad = 0 \tag{2.7l$'$}$$

and define three pairs of unitarity triangles, 6 in total:

- (2.7h$'$) and (2.7k$'$) are the ones shown in figure 2.1 with three sides of similar length, all of order $A\lambda^3$. This is "**the unitarity triangle**". The other ones are quite flat, and it will require very high precision to prove experimentally that they are not degenerate to a line. They are all shown to approximate scale in figure 2.2.
- (2.7i$'$) and (2.7l$'$) have two sides of length $A\lambda^2$ and one much shorter of order $A\lambda^4$. This limits the small angles, which are $\beta_s = \tilde{\phi}_2 + \tilde{\phi}_6$ and $\tilde{\phi}_2 - \tilde{\phi}_6$, respectively. They are close to the differences of angles in the large triangles $\gamma - \gamma' = \beta' - \beta = \tilde{\phi}_2 - \tilde{\phi}_4$.

The other two angles are for (2.7i′) $\sim \beta$ and $\sim \pi - \beta$, and for (2.7l′) $\sim \gamma$ and $\sim \pi - \gamma$.

- (2.7g′) and (2.7j′) have two sides of length λ and one very much shorter of order $A^2\lambda^5$, with a small angle $\tilde{\phi}_4 - \tilde{\phi}_6$ and $\tilde{\phi}_4 + \tilde{\phi}_6$, respectively. Both are of order λ^4.

  The other two angles are for (2.7j′) $\sim \beta$ and $\sim \pi - \beta$, and for (2.7g′) $\sim \gamma$ and $\sim \pi - \gamma$.

Tiny differences between the two standard unitarity triangles are $\mathcal{O}(\lambda^2)$ corrections,

$$\begin{array}{rcccl} A\lambda^3(1-\rho-i\eta) & + & -A\lambda^3 & + & A\lambda^3(\rho+i\eta) \quad = 0 \\ +A\lambda^5(\rho+i\eta-\frac{1}{2}) & & +A\lambda^5(\frac{1}{2}-\rho-i\eta) & & +\mathcal{O}(\lambda^7) \\ +\mathcal{O}(\lambda^7) & & +\mathcal{O}(\lambda^7) & & \end{array} \tag{2.7h''}$$

$$\begin{array}{rcccl} A\lambda^3(\rho+i\eta) & + & -A\lambda^3 & + & A\lambda^3(1-\rho-i\eta) = 0 \\ -\frac{1}{2}A\lambda^5(\rho+i\eta) & & +\mathcal{O}(\lambda^7) & & +\frac{1}{2}A\lambda^5(\rho+i\eta) \\ +\mathcal{O}(\lambda^7) & & & & +\mathcal{O}(\lambda^7) \end{array} \tag{2.7k''}$$

The angles in these two triangles can be estimated from experimental constraints on a 3×3 unitary CKM matrix element magnitudes, and directly measured in CP violation in B meson decays. However, deviations from or extensions to the Standard Model may imply that the two triangles are dissimilar, or even that they are no closed triangles at all. Therefore, it is important to distinguish measurements of different parameters, even if they are expected to have identical or close values within the three family Standard Model.

2.2 Phases and Observables

The fact that phases of quark fields are unobservable numbers has been used to show that phases in the CKM matrix are not observables either, and there remains some arbitrariness in the parametrization for this matrix. Any valid CKM matrix is obtained from (2.9) with five independent arbitrary phase angles $\zeta_1 \ldots \zeta_5$ as

$$\mathbf{V} = \begin{pmatrix} |V_{ud}|e^{i\zeta_1} & |V_{us}|e^{i(\zeta_1+\zeta_2)} & |V_{ub}|e^{i(\zeta_1+\zeta_3-\tilde{\gamma})} \\ -|V_{cd}|e^{i(\zeta_4+\tilde{\phi}_4)} & |V_{cs}|e^{i(\zeta_4+\zeta_2-\tilde{\phi}_6)} & |V_{cb}|e^{i(\zeta_4+\zeta_3)} \\ |V_{td}|e^{i(\zeta_5-\tilde{\beta})} & -|V_{ts}|e^{i(\zeta_5+\zeta_2+\tilde{\phi}_2)} & |V_{tb}|e^{i(\zeta_5+\zeta_3)} \end{pmatrix} \tag{2.10}$$

The freedom to choose quark phases may be extended to antiquarks, with six more phases $\bar{\phi}_u, \bar{\phi}_c, \bar{\phi}_t, \bar{\phi}_d, \bar{\phi}_s, \bar{\phi}_b$. With the new quark states

$$q'_j = e^{i\phi_j} q, \;\; \bar{q}'_j = e^{i\bar{\phi}_j} \bar{q}_j, \;\; j = u, c, t, d, s, b$$

also the phase induced by the CP operation is changed. The transition

$$\mathrm{CP}\,|q_j\rangle = e^{i\phi_{\mathrm{CP}\,j}}|\bar{q}_j\rangle \quad \rightarrow \quad \mathrm{CP}\,|q'_j\rangle = e^{i\phi'_{\mathrm{CP}\,j}}|\bar{q}'_j\rangle$$

requires

$$\phi'_{\mathrm{CP}\,j} = \phi_{\mathrm{CP}\,j} + \phi_j - \bar{\phi}_j$$

This equation leaves $\phi'_{\mathrm{CP}\,j}$ still completely undefined, since all three phases on the right-hand side are not observable, and therefore subject to arbitrary changes. It becomes meaningful, however, if it is applied to observables, like CP eigenvalues. Two CP eigenstates constructed from a meson and antimeson state with eigenvalues ± 1 are related accordingly:

$$|q_j\bar{q}_k\rangle \pm e^{i\phi_{\mathrm{CP}\,jk}}|q_k\bar{q}_j\rangle \;\; = \;\; e^{-i(\phi_j+\bar{\phi}_k)}\left[|q'_j\bar{q}'_k\rangle \pm e^{i\phi'_{\mathrm{CP}\,jk}}|q'_k\bar{q}'_j\rangle\right]$$

The new states $|q'_j\bar{q}'_k\rangle \pm e^{i\phi'_{\mathrm{CP}\,jk}}|q'_k\bar{q}'_j\rangle$ have the same eigenvalues, and differ by an overall unobservable phase from the old ones.

The CP operation on a meson, e.g. the pseudoscalar B^0 meson $|\bar{b}d\rangle$, is

$$\mathrm{CP}\,|B^0\rangle = e^{i\phi_{\mathrm{CP}B}}|\overline{B}^0\rangle \tag{2.11}$$

where the phase factor $e^{i\phi_{\mathrm{CP}B}} = \langle\overline{B}^0|\,\mathrm{CP}\,|B^0\rangle$ depends on the parity of the bound-state wave function, and the chosen quark and antiquark phase convention.

Quark phase changes could be compensated by phase changes of the CKM matrix elements according to (2.2), leaving terms like

$$\langle q_j|V_{jk}|q_k\rangle$$

invariant. However, the phase of this matrix element is **not** an observable. Hence the choice of phases in the CKM matrix parametrization can be made **independent** of the choice of quark phases.

Phase conventions will also enter into relations among decay amplitudes. An amplitude for a weak decay $B^0 \to X$ via a single well defined process can be written as

$$A = \langle X|\mathcal{H}|B^0\rangle = \langle X|\boldsymbol{O}V|B^0\rangle \tag{2.12}$$

where V is a product of the appropriate CKM matrix elements and $\boldsymbol{O}$ is an operator describing the rest of the weak and possibly also subsequent strong interaction processes involved in the transition. Since strong interaction (also weak interaction except for nontrivial phases in V) are CP invariant, the charge conjugate mirror process $\overline{B}^0 \to \overline{X}$ has an amplitude

$$\begin{aligned}\overline{A} = \langle\overline{X}|\mathcal{H}|\overline{B}^0\rangle &= \langle\overline{X}|\,\mathrm{CP}^+\,\mathrm{CP}\,\mathcal{H}\,\mathrm{CP}^+\,\mathrm{CP}\,|\overline{B}^0\rangle \\ &= e^{i\phi_{\mathrm{CP}X}}\langle X|\,\mathrm{CP}\,\boldsymbol{O}V\,\mathrm{CP}^+\,e^{-i\phi_{\mathrm{CP}B}}|B^0\rangle \\ &= e^{i(\phi_{\mathrm{CP}X}-\phi_{\mathrm{CP}B})}\langle X|\boldsymbol{O}V^*|B^0\rangle \\ &= e^{i(\phi_{\mathrm{CP}X}-\phi_{\mathrm{CP}B})}\frac{V^*}{V}A\end{aligned} \tag{2.13}$$

where also

$$\frac{V^*}{V} = e^{-2i\arg V}$$

is just a phase. Especially, if X is a CP eigenstate with eigenvalue $\xi_X = \pm 1$,

$$\overline{A} = \xi_X e^{-i(\phi_{\mathrm{CP}B}+2\arg V)}A \tag{2.14}$$

relates the two amplitudes, and the ratio $\overline{A}/A$ flips sign with the CP eigenvalue.

All physical observables must be independent of the choice of phases. This is the case if only absolute values of amplitudes are involved, but for interference terms the phase convention cancels often in a more subtle way. One example is the ratio

$$\lambda_{\mathrm{CPV}} := \frac{q\overline{A}}{pA}$$

where the $e^{i\phi_{\mathrm{CP}B}}$ factors cancel from

$$\frac{q}{p} = \frac{\langle\overline{B}^0|B_L\rangle}{\langle B^0|B_L\rangle} = e^{i\phi_{\mathrm{CP}B}}\dots$$

and only rephasing invariant products of CKM matrix elements remain.

On the other hand, expressions where the arbitrary phases are still present cannot be observables.

2.3 Reasonable Phase Conventions

Once the distinction of unobservable and observable phases is clear, it is reasonable to choose phases in a way that simplifies calculations.

So it is sensible to use only one phase in the CKM matrix as in (2.3), and not six as in (2.10). If a choice of phases were possible where all CKM matrix elements can be made real, also charged current weak interactions would not violate CP symmetry.

A natural choice for CP phases requires all $J^{PC} = 0^{-+}$ mesons to have CP $|X\rangle = -|\overline{X}\rangle$, fixing $\phi_{\mathrm{CP}B} = \pi$. However, it has become fashionable to use the opposite sign convention, i. e. $\phi_{\mathrm{CP}B} = 0$.

The appearance of an additional phase factor in $e^{i\phi_{\mathrm{CP}\,kj}}\langle\bar{q}_j|V^*_{jk}|\bar{q}_k\rangle$ can be avoided by the restriction $\bar{\phi}_j = -\phi_j$ for quark phase changes, and an appropriate phase convention which makes terms related by a CPT transformation relatively real.

2.4 An Example of a Different Phase Convention

The standard phase convention (2.9)

$$V = \begin{pmatrix} |V_{ud}| & |V_{us}| & |V_{ub}|e^{-i\tilde{\gamma}} \\ -|V_{cd}|e^{i\tilde{\phi}_4} & |V_{cs}|e^{-i\tilde{\phi}_6} & |V_{cb}| \\ |V_{td}|e^{-i\tilde{\beta}} & -|V_{ts}|e^{i\tilde{\phi}_2} & |V_{tb}| \end{pmatrix}$$

puts large phases into the smallest CKM matrix elements, V_{ub} and V_{td}. A phase transformation of the u and d quark leads immediately to another phase convention

$$V = \begin{pmatrix} -|V_{ud}|e^{-i\alpha} & |V_{us}|e^{i\tilde{\gamma}} & |V_{ub}| \\ -|V_{cd}|e^{i(\tilde{\phi}_4+\tilde{\beta})} & |V_{cs}|e^{-i\tilde{\phi}_6} & |V_{cb}| \\ |V_{td}| & -|V_{ts}|e^{i\tilde{\phi}_2} & |V_{tb}| \end{pmatrix} \tag{2.15}$$

where $\alpha = \pi - \tilde{\beta} - \tilde{\gamma}$ has been used to simplify the phase of V_{ud}. In this representation big phases are associated with V_{ud}, V_{us}, and V_{cd}, while V_{ub} and V_{td} are real.

For the CP violating observable $\sin 2\beta$, the asymmetry of a pure $b\bar{d} \to c\bar{c}d\bar{d}$ decay and realized almost perfectly in $B^0 \to J/\psi K^0_S \to J/\psi\pi\pi$, the relevant phase is $\beta = \arg(-V^*_{td}V_{tb}V_{cd}V^*_{cb})$. This is a rephasing-independent definition, and can be depicted via a unitarity triangle.

The standard phase convention (2.9) puts most of the phase β into V_{td}, which contributes via the box diagram of $B^0\overline{B}^0$ mixing to the asymmetry. This tempts the experimentalist to call β the "mixing phase" of the $B^0 = B_d$ system.

The alternative phase convention (2.15), however, puts it into V_{cd}, which enters into the decay amplitude, here β is a "decay phase" while the "mixing phase" is 0 in this parametrization!

Similarly, the pure tree $b\bar{d} \to u\bar{u}d\bar{d}$, which can be disentangled from $B \to \pi\pi$ final states using an isospin analysis, has a CP aymmetry given by the rephasing-independent $\alpha = \arg(-V^*_{td}V_{tb}V_{ud}V^*_{ub})$.

The standard phase convention (2.9) puts this phase partly into V_{td} ($\tilde{\beta}$ entering via mixing) and partly into V_{ub} ($\tilde{\gamma}$ entering via decay), resulting in $\alpha = \pi - \tilde{\beta} - \tilde{\gamma}$.

The alternative phase convention (2.15), however, puts it into V_{ud}, which enters only into the decay amplitude as $\alpha = \arg V_{ud}$.

In any case, it is equivalent to "measure α" and to "measure $\beta + \gamma$", since $\alpha + \beta + \gamma = \pi$ holds in any parametrization of the CKM matrix.

In the same way it is possible to make V_{ts} real, via an additional phase transformation of the s quark, leading to another valid parametrization of the CKM matrix

$$V = \begin{pmatrix} -|V_{ud}|e^{-i\alpha} & |V_{us}|e^{i(\tilde{\gamma}-\tilde{\phi}_2)} & |V_{ub}| \\ -|V_{cd}|e^{i(\tilde{\phi}_4+\tilde{\beta})} & |V_{cs}|e^{-i(\tilde{\phi}_2+\tilde{\phi}_6)} & |V_{cb}| \\ |V_{td}| & -|V_{ts}| & |V_{tb}| \end{pmatrix} \tag{2.16}$$

so there is no “mixing phase” for the B_s^0 meson either.

3. Conclusions

An observable phase can only be associated with a product of at least four CKM matrix elements, the phase of a product of less than four is always arbitrary. When we measure the CP violating phases β or β_s of the unitarity triangles, we should never call them “mixing phases” or “mixing angles”, because they are always a combination of mixing and decay phases, and it is in principle impossible to disentangle these contributions.

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